∞-Cats Reading group 10/29/2022.

Invertible natural transfo's + ∞-cats as fibrant objects.
(§3.5) (§3.6)

§3.5: Invertible nat'l transfo's

Recall: Joyal's thm (3.4.18)

Let $p: X \to Y$ be an inner fib. btwn ∞-cats.
Given a $\Lambda^n_0 \to X$ s.t. the "first edge is invertible in X"

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r] \arrow[d] & J = \{0 \overset{\sim}{\underset{\sim}{\rightleftarrows}} 1\} \arrow[d] \\ \Lambda^n_0 \arrow[r] & X \end{tikzcd}$$

then there's a lift in any diagram of the form

$$\begin{tikzcd} \Lambda^n_0 \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}.$$

"Kan Cpxes & Quasicats"
– Joyal

Cor (3.5.1) (Joyal) Kan cpxes are the same as ∞-gpds.

any horn fills

every edge is invertible + inner horn filling

Pf: $\Rightarrow$: Show inverses by solving appropriate (outer) horn-filling problems.

$\Leftarrow$: Use Joyal's thm to fill in outer horns. $\blacksquare$

Def. Let Gpd = {small gpds} $\subseteq$ Cat.

There are adjoints

$$\pi_1 = \text{gpfy} \left(\begin{array}{c} \text{Gpd} \\ \dashv \downarrow \dashv \\ \text{Cat} \end{array} \right) (-)^{\simeq} = k(-)$$

$\text{gpdfy}(C) := C[\text{Hom}(C)^{-1}]$

$C^{\simeq} := \{\text{all equiv's of } C\} \subseteq C$.

$\boxed{\text{core}}$

Can extend this to ∞-land. Let $\mathcal{C} \in$ qCat. Form the pback:

$$\begin{tikzcd} \mathcal{C}^{\simeq} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C} \arrow[d] \\ N((h\mathcal{C})^{\simeq}) \arrow[r] & N(h\mathcal{C}) \end{tikzcd}.$$

Similarly there's an adjunction

$$\begin{array}{c} \text{Kan} = \infty\text{-gpds} \\ \downarrow \dashv \uparrow (-)^{\simeq} \\ \text{qCat} \end{array}.$$

In particular, $\mathcal{C}^{\simeq} \subseteq \mathcal{C}$ is the largest Kan cpx living inside $\mathcal{C}$. / maximal ∞-gpd.

Prop (3.5.5) Given a left/right fibration $p: X \to Y$

if Y is a Kan cpx. $\Rightarrow$ then so is X
& p is a Kan fibration.

$$\begin{tikzcd} \Lambda^n_0 \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur] & Y \end{tikzcd} \qquad \begin{tikzcd} \Lambda^{n}_{all} \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta \arrow[r] & Y \end{tikzcd}$$

Pf. (i) A left/right is conservative $\Rightarrow X$ is a Kan cpx.
(3.4.8)

"lifts equivs"
$p(f)$ is invertible in Y
$\Rightarrow f$ invertible in X

(ii) Want
$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur] & Y \end{tikzcd}$$
. When $n \geq 2$ can use Joyal's thm + Kan cpx.

When $n = 1$ ...
$$\begin{tikzcd} \Lambda^1_0 \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur] & Y \end{tikzcd} \qquad \begin{tikzcd} x' \arrow[r, dashed] \arrow[d, no head, squiggly] & x \arrow[d, no head, squiggly] \\ p(x') = y \arrow[r, "\sim"] & p(x) \end{tikzcd}$$

this is an eq. in Y (Kan cpx)
& left/right fibs btwn ∞-cats
$\Rightarrow$ isofibrations.
$\to$ can lift such equiv's. $\boxtimes$

Cor Given a L/R fib. $p: X \to Y$, then for any $y \in Y_0$, the fiber X_y ... is a Kan complex.

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "p"] \\ \Delta^0 \arrow[r, "y"'] & Y \end{tikzcd}$$

Pf. Use previous prop on $X_y \to \Delta^0$. $\boxtimes$

$X, Y \in$ sSet
$\underline{\mathrm{Hom}}(X,Y) =$ "function cpx"
$\in$ sSet.
when $Y \in q\mathcal{C}at$,
$\underline{\mathrm{Hom}}(X,Y) \in q\mathcal{C}at$.

Def. Let $A \in$ sSet $A_0 =$ constant sset of objects of A.
$X \in q\mathcal{C}at$ $\subseteq A$.

Define
$$\begin{tikzcd} k(A,X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \underline{\mathrm{Hom}}(A,X) \arrow[d] \\ \underline{\mathrm{Hom}}(A_0,X)^{\simeq} \arrow[r] & \underline{\mathrm{Hom}}(A_0,X) \end{tikzcd}$$

ie. $k(A,X)$ — (0): maps $A \xrightarrow{F} X$
— (1): nat'l transfo's $\phi: F \to F'$
s.t. $\forall a \in A_0$ the map $\phi_a: F(a) \to F'(a)$ is invertible in X.

ie. $k(A,X) = \{$ptwise invertible nat'l transfo's$\}$.

Rk. Notice that $\underline{\mathrm{Hom}}(A,X)^{\simeq} \hookrightarrow k(A,X)$ since invertible nat'l transfos $\Rightarrow$ ptwise invertible.
" " $\Leftarrow$ " "

Prop (3.5.12): In fact, $\underline{\mathrm{Hom}}(A,X)^{\simeq} = k(A,X)$.

More generally for • an isofib. $p: X \to Y$ btwn ∞-cats.
• a mono. $A \to B$

then
$$k(A,X) \underset{k(A,Y)}{\times} k(B,Y) = \left[\underline{\mathrm{Hom}}(A,X) \underset{\underline{\mathrm{Hom}}(A,Y)}{\times} \underline{\mathrm{Hom}}(B,Y)\right]^{\simeq}$$

Pf. [skipped]

Def. $A \in sSet$, $X \in q\mathcal{C}at$

$h(A, X) \subseteq \underline{Hom}(A, X)$ —— $h(A,X)_n = \{ A \to \underline{Hom}(\Delta^n, X)$ that factor as… $\}$, via $k(\Delta^n, X) \hookrightarrow \underline{Hom}(\Delta^n, X)$.

(n) $A \times \Delta^n \to X$
$\Updownarrow$
$A \to \underline{Hom}(\Delta^n, X)$

i.e. $h(A,X) = \{$ maps $A \xrightarrow{F} X$ s.t. F inverts everything in $A \}$

Note: When $A = J$, $h(J, X) = \underline{Hom}(J, X)$

$A = \Delta^0$, $h(\Delta^0, X) = \underline{Hom}(\Delta^0, X) \cong X$.

Prop. (3.5.13) Given
- an isofib. $X \xrightarrow{p} Y$ btwn ∞-cats
- an anodyne ext. $A \to B$

there's a trivial fibration

$$h(B, X) \longrightarrow h(A,X) \times_{h(A,Y)} h(B,Y).$$

---

§3.6: ∞-cats as fibrant objects.

Recall… Joyal's model structure on ssets.

$W = \{$weak cat'l equivalences$\}$

$S \xrightarrow{f} T$ …. $\forall\, \mathcal{C} \in q\mathcal{C}at$, there's a map $\underline{Hom}(T, \mathcal{C}) \xrightarrow{f^*} \underline{Hom}(S, \mathcal{C})$

$\rightsquigarrow \pi_0$

$\pi_0 \underline{Hom}(T, \mathcal{C}) \xrightarrow{\sim} \pi_0 \underline{Hom}(S, \mathcal{C})$ a bijection.

cof = monomorphisms

fib $\supseteq$ isofibrations

$X \xrightarrow{p} Y$ s.t.

$$\begin{tikzcd} \Delta^{\{0\}} \arrow[r] \arrow[d] & X \arrow[d] \\ J \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Note: in Cisinski isofibrations = inner fibs w/ such a map.
elsewhere isofibrations = lifting property.

cf. the "canonical model structure" on Cat

W = equiv's of cats.

cof = functors injective on objects

fib = (ordinary) isofibrations

Thm (3.6.1) (Joyal): A sset is fibrant in $sSet_{Joyal}$ iff it's an ∞-cat. [HTT, 2.4.6.1]

Fibrations in $sSet_{Joyal}$ are precisely isofibrations (Cisinski sense). [HTT, 2.4.6.5]

Pf. (skipped)

Prop (3.6.2) : W_{Joyal} is the smallest class of maps in sSet satisfying...

(1) 2-out-of-3
(2) Any inner anodyne ext. is in W_{Joyal}
(3) Any trivial fib. btwn ∞-cats is in W_{Joyal}.

Cor (3.6.6) : A functor btwn ∞-cats $f : \mathcal{C} \to \mathcal{D}$

$$\left(\text{is an equiv. of } \infty\text{-cats}\right) \iff \left(\begin{array}{l}\text{it's a weak eq. in } sSet_{Joyal} \\ \text{ie. weak nat'l equiv.}\end{array}\right)$$

$\exists g : \mathcal{D} \to \mathcal{C}$
& inv. nat'l transf.
$fg \Rightarrow id_{\mathcal{D}}$
$gf \Leftarrow id_{\mathcal{C}}$

Rk (3.6.5) Invertible nat'l transfos are invertible edges in $\underline{Hom}(X,Y)$ Δ^1

ie. edges

$$\begin{tikzcd} \Delta^1 \arrow[r] \arrow[dr] & \underline{Hom}(X,Y) \\ & J \arrow[u] \end{tikzcd}$$

$$Hom_{sSet}(J, \underline{Hom}(X,Y)) \cong Hom_{sSet}(J \times X, Y)$$

ie. two maps of ssets are "related by inv. nat'l transf." iff they're "J-htpic"

ie. equiv.'s of ∞-cats are J-htpy equiv's ie. w-equiv's in $sSet_{Joyal}$.

Thm (3.6.8) Let $f : X \to Y$ be a map of ssets. TFAE..

(1) f is a weak categorical equivalence. (ie. w.e.)

(2) $\forall$ ∞-cat $\mathcal{C}$, there's an equiv. (of ∞-cats)

$$\underline{Hom}(Y,\mathcal{C}) \xrightarrow{\sim} \underline{Hom}(X,\mathcal{C}).$$

(3) " " there's an equiv. (of ordinary cats)

$$h\underline{Hom}(Y,\mathcal{C}) \xrightarrow{\sim} h\underline{Hom}(X,\mathcal{C})$$

(4) " " , there's an equivalence (of ∞-gpds)

$$\begin{array}{ccc} k(Y,\mathcal{C}) & \xrightarrow{\sim} & k(X,\mathcal{C}) \\ \| & & \| \\ \underline{Hom}(Y,\mathcal{C})^{\simeq} & & \underline{Hom}(X,\mathcal{C})^{\simeq} \end{array}$$

<u>Pf.</u> (1) $\iff$ (2)

$\Rightarrow$: Say f is a weak cat'l equiv. $\iff$ $\forall$ ∞-cat $\mathcal{C}$ (fibrant object), there's a bijection

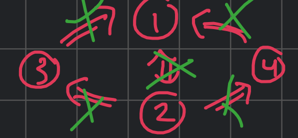
$$[Y,\mathcal{C}]_J \cong [X,\mathcal{C}]_J$$

$[A,B]_J = Hom_{sSet}(A,B)/\sim$ J-htpy

By our discussion that's the same as equiv. of cats.

$\Leftarrow$: If $\underline{Hom}(Y,\mathcal{C}) \xrightarrow{\sim} \underline{Hom}(X,\mathcal{C})$

$$\begin{array}{c} \underline{Hom}(Y,\mathcal{C}) \xrightarrow{\sim} \underline{Hom}(X,\mathcal{C}) \\ \big\}\ \pi_0 \\ \pi_0\underline{Hom}(Y,\mathcal{C}) \xrightarrow{\sim} \pi_0\underline{Hom}(X,\mathcal{C}) \end{array} \Rightarrow f \text{ is a weak cat'l eq.}$$



$(2) \Rightarrow (3)$: Easy. //

$(3) \Rightarrow (1)$: $h\underline{\mathrm{Hom}}(y,\mathcal{C}) \xrightarrow{\simeq} h\underline{\mathrm{Hom}}(x,\mathcal{C})$

$\Rightarrow$ a bijection on objects

$\pi_0 \underline{\mathrm{Hom}}(y,\mathcal{C}) \cong \pi_0 \underline{\mathrm{Hom}}(x,\mathcal{C})$. $\Rightarrow f$ is a weak-cat'l equiv.

$(2) \Rightarrow (4)$: $\underline{\mathrm{Hom}}(y,\mathcal{C}) \xrightarrow{\sim} \underline{\mathrm{Hom}}(x,\mathcal{C}) \rightsquigarrow k(y,\mathcal{C}) \xrightarrow{\sim} k(x,\mathcal{C})$

$$k(y,\mathcal{C}) = \underline{\mathrm{Hom}}(y,\mathcal{C})^{\simeq}, \quad k(x,\mathcal{C}) = \underline{\mathrm{Hom}}(x,\mathcal{C})^{\simeq}$$

WTS: $(-)^{\simeq} : sSet \to sSet$

sends weak eq's btwn ∞-cats $\mapsto$ eq's of Kan cpxes.

By Ken Brown's lemma we can show this is true if $(-)^{\simeq}$ sends trivial fibs $\mapsto$ trivial fibs.

$$sSet \xrightarrow{(-)^{\simeq}} sSet$$

a trivial fib. btwn. ∞-cats. (fib. obj's)

$$\begin{tikzcd} X \arrow[d, "p"] \\ Y \end{tikzcd} \longmapsto \begin{tikzcd} X^{\simeq} \arrow[d, "p|_{X^{\simeq}}"] \\ Y^{\simeq} \end{tikzcd}$$

This map $p|_{X^{\simeq}}$ sits in a square

$$\begin{tikzcd} X^{\simeq} \arrow[r, hook] \arrow[d, "p^{\simeq}"'] & X \arrow[d, "p"] \\ Y^{\simeq} \arrow[r, hook] & Y \end{tikzcd}$$

WTS: $p^{\simeq} = p|_{X^{\simeq}}$ is a triv. fib.

$\Updownarrow$

the square is a pback $\left(X^{\simeq} \cong X \times_Y Y^{\simeq}\right)$ (maps sent to weak eq's in Y)

$\subseteq$: p sends eq's $\mapsto$ eq's.

$\supseteq$: p conservative.

$(4) \Rightarrow (1)$: An equiv. of ∞-gpds $\underline{\mathrm{Hom}}(y,\mathcal{C})^{\simeq} \xrightarrow{\sim} \underline{\mathrm{Hom}}(x,\mathcal{C})^{\simeq}$

$\Rightarrow$ a bijection $\pi_0(\ \text{''}\ ) \xrightarrow{\sim} \pi_0(\ \text{''}\ )$ — by def.

$$\mathrm{Hom}_{sSet}(y,\mathcal{C})/\text{inv. nat'l transf.} \cong \mathrm{Hom}_{sSet}(x,\mathcal{C})/\text{inv. nat'l transf.}$$

$$[y,\mathcal{C}]_J \cong [x,\mathcal{C}]_J$$ — by 3.6.5.

$\Rightarrow f$ is a weak. cat'l equiv. ☒

Prop (1.6.1..) similar prop.